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ABSTRACT

Formulas for the cross-correlation and spectral density functions of the scalar wave field radiated by a random point source in a three-dimensional time-dependent random medium are derived. The medium is assumed to be statistically homogeneous and isotropic and to be statistically independent of the source. The analysis is based on a modification of the smoothing method. An approximate expression for the power spectrum of the wave as a function of the source-field point distance (or propagation distance) is obtained for the case in which the characteristic frequency of the source is much greater than that of the medium. This expression shows that the wave spectrum approaches a limiting form, which is referred to here as the fully-developed spectrum, with increasing propagation distance. It is also found that the total signal power is conserved as the spectrum evolves. Results obtained for the case of a narrow-band source indicate that the spectral bandwidth increases initially as the square root of the propagation distance, but that at larger distances it approaches a limiting value. Numerical results obtained for the narrow-band case show a progressive broadening of the wave spectrum with increasing propagation distance and/or with increasing strength of the randomness of the medium, in agreement with observations.

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INTRODUCTION

Broadening of the frequency spectrum of an initially narrow-band wave field is a phenomenon which is characteristic of wave propagation in a time-dependent medium, and is a result of amplitude and frequency modulation of the spectral components of the wave by the time variations in the properties of the medium. Of particular practical interest is the effect of <u>random</u> fluctuations of the medium, and indeed spectral broadening due to propagation through turbulence has been observed in the case of both acoustic and electromagnetic waves [1,2]. The present investigation was undertaken with the purpose of studying this effect, i.e., spectral broadening arising from the presence of random, time-dependent fluctuations of the medium, from a rather general point of view.

Previous theoretical investigations of spectral broadening of waves in random media include those of Howe[3], Fante [4,5], and Woo, *et al*.[6]. Howe derived a kinetic equation and used it to study the effect of the random velocity field on the frequency spectrum of an acoustic wave propagating in a turbulent fluid. Fante used transport theory to study frequency spectra of beamed waves propagating in a turbulent atmosphere. The analysis of Woo, *et al*. (see also Ref. 7, p. 422) is based on the parabolic approximation. Howe treated the case of an isotropic time-dependent turbulence field, whereas both Fante and Woo, *et*. *al*. assumed that the time variations of the medium were the result of a steady mean wind convecting a "frozen" turbulence field in a direction perpendicular to the direction of propagation.

The authors mentioned above based their analyses on different mathematical models and/or calculated different statistical properties of the wave field than those considered in the present investigation (e.g., Woo, *et al*. considered the spectrum of the complex wave amplitude, whereas the present investigation deals with the entire wave function, which is a real quantity), and hence their results do not agree in all respects with those obtained here. The results of both Howe and Fante indicate that, over a suitably restricted propagation path and for high-frequency waves, the characteristic width of the wave spectrum increases as some power of the propagation distance. The results of Woo, et al. are given in a more complicated form, but seem to show a similar effect. These results agree generally with those obtained here for small propagation distances. At large distances, however, the present results indicate that the spectral width approaches a limiting value, which is not predicted by any of the theories mentioned above.

The problem of spectral broadening has also been treated in a recent paper by Kuznetsova and Chernov[8]. Their analysis, like that of Woo, et al., is based on the parabolic approximation with a frozen turbulence model. Their results also indicate an increase of the spectral width as some power of the propagation distance for small propagation distances. However, since their expression for the wave spectrum is given in the form of a power series in the propagation distance, the behavior of the spectrum for large propagation distances, as predicted by their theory, is not clear.

Spectral broadening in a random medium has also been discussed from a theoretical viewpoint by Adomian[9]; however, that author did not obtain an explicit analytical expression for the wave spectrum. Related work, concerned mainly with spectra of scattered waves (in contrast to the present investigation, which deals with the spectrum of the total wave field) and with spectra of amplitude and phase fluctuations of waves propagating in random media, can be found in Refs. 10-16.

I. ANALYSIS

The starting point of the analysis is the scalar wave equation

$$(c^{-2}\,\partial_t^2 - \nabla^2)u = f \quad , \tag{1}$$

where u is the wave function, f is the source term, and c is the local propagation speed of small disturbances of the medium. All quantities are assumed to be real functions of t and $\underset{\sim}{x}$, where t is time and $\underset{\sim}{x}$ $[=(x_1, x_2, x_3)]$ is a three-dimensional spatial coordinate.

The propagation speed c is assumed to be random; i.e., c is assumed to depend on a parameter a which is an element of a sample space A. The space A, together with a σ-algebra of subsets and a probability measure, forms a probability space. The source term f is also random; however f is assumed to be statistically independent of c. Thus, f may be regarded as being dependent on a parameter b ranging over a different sample space B which, together with its own σ-algebra of subsets and probability measure, also forms a probability space.

It is clear that the solution u of Equation 1, as well as functions of it, will depend on both a and b. (The dependence on the parameters a and b of the various quantities appearing in the analysis will not, in general, be explicitly indicated.) It will be necessary, therefore, in what follows to distinguish between ensemble averages over the space A, which will be denoted by $\langle\ \rangle_A$, and averages over B, denoted by $\langle\ \rangle_B$. An average over both A and B (i.e., an ensemble average over the product sample space $A \times B$) will be denoted simply by $\langle\ \rangle$. We note that generally $\langle\ \rangle = \langle\langle\ \rangle_A\rangle_B = \langle\langle\ \rangle_B\rangle_A$.

In most cases involving wave propagation in real media such as the atmoshere or ocean the fluctuations in the medium properties can be regarded as small.

Thus it is realistic, as well as mathematically convenient, to write c in the form

$$c(t,x) = c_0[1+\varepsilon\mu(t,\underset{\sim}{x})] \quad . \tag{2}$$

Here ε is a small parameter which is a measure of the magnitude of the fluctuations of the medium, and μ is a random function with zero mean and unit variance; i.e., $\langle\mu\rangle_A = 0$, $\langle\mu^2\rangle_A = 1$. The quantity c_0, the average of c, is assumed to be a constant.

Writing c as in Equation 2 allows the problem to be solved by a perturbation technique. To begin, we substitute the expression for c given by Equation 2 into Equation 1 and expand in powers of ε. This yields

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2 + \cdots)u = f \quad , \tag{3}$$

where the operators L_0, L_1, and L_2 are given by

$$L_0 = c_0^{-2}\partial_t^2 - \nabla^2, \tag{4}$$

$$L_1 = -2c_0^{-2}\mu\partial_t^2 \quad , \tag{5}$$

$$L_2 = 3c_0^{-2}\mu^2\partial_t^2 \quad . \tag{6}$$

From Equation 3, approximate equations, valid when ε is small, can be obtained for $\bar{u}$ and $\tilde{u}$, where $\bar{u} \equiv \langle u\rangle_A$ and $\tilde{u} \equiv u - \langle u\rangle_A$. The procedure is entirely analogous to that described by Keller[17] (see also Ref. 18). It is only necessary to keep in mind that since f is independent of a it is unaffected by averaging over A. When $\langle L_1 \rangle_A = 0$, which is usually the case in practice, these equations reduce to

$$M\bar{u} = f \quad , \tag{7}$$

$$\tilde{u} = -\varepsilon L_0^{-1} L_1 \bar{u} \quad , \tag{8}$$

where the operator M is defined by

$$M = L_0 + \varepsilon^2 (\langle L_2 \rangle_A - \langle L_1 L_0^{-1} L_1 \rangle_A) \quad . \tag{9}$$

Terms of order ε^3 have been dropped from Equation 7; terms of order ε^2 have been dropped from Equation 8.

In the special case in which f is determinate (i.e., non-random) the quantities $\bar{u}$ (which is then also determinate) and $\tilde{u}$ correspond respectively to the mean and fluctuating fields. In that context the type of approach leading to Equations 7 and 8, which involves obtaining separate equations for the mean and fluctuating fields, is referred to as the smoothing method by Frisch[18].

We shall be concerned in the remainder of this paper only with random processes which are stationary in time. (By stationary we mean stationary in the wide sense, i.e., that correlation functions of the form given by Equation 19 are independent of t.) In order to ensure that $u(t,\underset{\sim}{x})$ is stationary in time we shall assume that both $\mu(t,\underset{\sim}{x})$ and $f(t,\underset{\sim}{x})$ are stationary in time. That these assumptions are sufficient for our purposes will become clear as the analysis proceeds. In addition, we shall assume for convenience that $\mu(t,\underset{\sim}{x})$ is statistically homogeneous and isotropic in space, and that $\langle f\rangle_B = 0$.

We introduce next the Green's functions $G_0(t,\underset{\sim}{x})$ and $G(t,\underset{\sim}{x})$, which are solutions of the equations

$$L_0 G_0(t,\underset{\sim}{x}) = \delta(t)\delta(\underset{\sim}{x}) \quad , \tag{10}$$

$$M\, G(t,\underset{\sim}{x}) = \delta(t)\delta(\underset{\sim}{x}) \quad , \tag{11}$$

and which satisfy the initial conditions $G_0 = G = 0$ for $t < 0$. (No boundary conditions need be imposed on G_0 or G since we are considering only free-space propagation.) Then L_0^{-1} can be written in the form

$$L_0^{-1} w(t,\underset{\sim}{x}) = \iint G_0(t-t',\underset{\sim}{x}-\underset{\sim}{x}')w(t',\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad , \tag{12}$$

where $w(t,x)$ is any function for which the integral exists. (Here, and henceforth, an integral sign without limits denotes an integral from $-\infty$ to $+\infty$.) Similarly, the solution of Equation 7 can be expressed as

$$\bar{u}(t,\underset{\sim}{x}) = \iint G(t-t',\underset{\sim}{x}-\underset{\sim}{x}')f(t',\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad . \tag{13}$$

By making a change of integration variable we can write Equations 12 and 13 in the form

$$L_0^{-1} w(t,\underset{\sim}{x}) = \iint G_0(t',\underset{\sim}{x}')w(t-t',\underset{\sim}{x}-\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad , \tag{14}$$

$$\bar{u}(t,\underset{\sim}{x}) = \iint G(t',\underset{\sim}{x}')f(t-t',\underset{\sim}{x}-\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad . \tag{15}$$

It should be pointed out that, as a consequence of the assumption that $\mu(t,\underset{\sim}{x})$ is stationary in t and $\underset{\sim}{x}$, the operator M commutes with both time and space translations. This allows the Green's function G in Equation 13 to be written as a function of the differences $t-t'$ and $\underset{\sim}{x}-\underset{\sim}{x}'$, instead of as a function of t,t', $\underset{\sim}{x}$, and $\underset{\sim}{x}'$ separately. Since the operator L_0 has constant coefficients, it also commutes with both time and space translations, and hence the Green's function G_0 in Equation 12 can also be written as a function of $t-t'$ and $\underset{\sim}{x}-\underset{\sim}{x}'$. That G_0 and G can be written as functions of $t-t'$ in Equations 12 and 13 is necessary for the stationarity of u. Note also that both G_0 and G are determinate functions.

Operating with L_1 on $\bar{u}$, as given by Equation 13, yields

$$L_1\bar{u}(t,\underset{\sim}{x}) = -2c_0^{-2}\iint \mu(t,\underset{\sim}{x})G_{tt}(t-t',\underset{\sim}{x}-\underset{\sim}{x}')f(t',\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad . \tag{16}$$

(The subscripts on G denote derivatives.) By making a change of integration variable we can write Equation 16 in the form

$$L_1\bar{u}(t,\underset{\sim}{x}) = -2c_0^{-2}\iint \mu(t,\underset{\sim}{x})G_{tt}(t',\underset{\sim}{x}')f(t-t',\underset{\sim}{x}-\underset{\sim}{x}')dt'd\underset{\sim}{x}' \quad . \tag{17}$$

Operating on Equation 17 with L_0^{-1}, as given by Equation 14, and substituting the result into Equation 8 yields

$$\tilde{u}(t,\underset{\sim}{x}) = 2\varepsilon c_0^{-2} \int\cdots\int G_0(t',\underset{\sim}{x}')G_{tt}(t'',\underset{\sim}{x}'')$$

$$\times\ \mu(t-t',\underset{\sim}{x}-\underset{\sim}{x}')f(t-t'-t'',\underset{\sim}{x}-\underset{\sim}{x}'-\underset{\sim}{x}'')dt'd\underset{\sim}{x}'dt''d\underset{\sim}{x}'' \quad . \tag{18}$$

The cross-correlation function $R(\tau,\underset{\sim}{x},\underset{\sim}{y})$ is defined by

$$R(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle u(t,\underset{\sim}{x})u(t-\tau,\underset{\sim}{y})\rangle \ . \tag{19}$$

Upon writing u as the sum $u = \bar{u} + \tilde{u}$ in Equation 19 we obtain

$$R(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle \bar{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})\rangle + \langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle + \langle \tilde{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})\rangle$$

$$+ \langle \tilde{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle \ . \tag{20}$$

The two cross terms on the right-hand side of Equation 20; i.e., the terms involving products of $\bar{u}$ and $\tilde{u}$, vanish. This follows from the fact that $\bar{u}$ is independent of a and that $\langle\tilde{u}\rangle_A = 0$. Thus, for the first cross term, we can write

$$\langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle = \langle\langle \bar{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle_A\rangle_B = \langle \bar{u}(t,\underset{\sim}{x})\langle\tilde{u}(t-\tau,\underset{\sim}{y})\rangle_A\rangle_B = 0$$

(since $\langle\tilde{u}(t-\tau,\underset{\sim}{y})\rangle_A = 0$), and similarly for the second cross term. Expressions for the remaining two terms on the right-hand side of Equation 20 can be obtained with the aid of Equations 15 and 18, after which Equation 19 can be written

$$R(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) + \tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) \quad , \tag{21}$$

where

$$\bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle \bar{u}(t,\underset{\sim}{x})\bar{u}(t-\tau,\underset{\sim}{y})\rangle$$

$$= \int\cdots\int G(t',\underset{\sim}{x}')G(s',\underset{\sim}{y}')R_0(\tau-t'+s',\underset{\sim}{x}-\underset{\sim}{x}',\underset{\sim}{y}-\underset{\sim}{y}')$$

$$\times\ dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}' \quad , \tag{22}$$

$$\tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle \tilde{u}(t,\underset{\sim}{x})\tilde{u}(t-\tau,\underset{\sim}{y})\rangle = 4\varepsilon^2 c_0^{-4}$$

$$\times \int\cdots\int G_0(t',\underset{\sim}{x}')G_0(s',\underset{\sim}{y}')G_{tt}(t'',\underset{\sim}{x}'')G_{tt}(s'',\underset{\sim}{y}'')$$

$$\times\ \Gamma(\tau-t'+s',\underset{\sim}{y}-\underset{\sim}{x}+\underset{\sim}{x}'-\underset{\sim}{y}')R_0(\tau-t'+s'-t''+s'',\underset{\sim}{x}-\underset{\sim}{x}'-\underset{\sim}{x}'',\underset{\sim}{y}-\underset{\sim}{y}'-\underset{\sim}{y}'')$$

$$\times\ dt'd\underset{\sim}{x}'ds'd\underset{\sim}{y}'dt''d\underset{\sim}{x}''ds''d\underset{\sim}{y}'' \quad . \tag{23}$$

The correlation functions Γ and R_0 are defined by

$$\Gamma(\tau,\underset{\sim}{\xi}) = \langle \mu(t,\underset{\sim}{x})\mu(t-\tau,\underset{\sim}{x}+\underset{\sim}{\xi})\rangle_A \quad , \tag{24}$$

$$R_0(\tau,\underset{\sim}{x},\underset{\sim}{y}) = \langle f(t,\underset{\sim}{x})f(t-\tau,\underset{\sim}{y})\rangle_B \quad . \tag{25}$$

In deriving Equations 22 and 23 use has been made of the fact that μ is independent of b and that f is independent of a.

Note that $u(t,\underset{\sim}{x})$, as calculated here, is indeed stationary in time, as can be seen by referring to Equation 19 and Equations 21-23.

The spectral density function $S(\omega,\underset{\sim}{x},\underset{\sim}{y})$ is defined by

$$S(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int R(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau \quad . \tag{26}$$

With the aid of Equation 21 we can write

$$S(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) + \tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) \quad , \tag{27}$$

where

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int \bar{R}(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau \quad , \tag{28}$$

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int \tilde{R}(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau \quad . \tag{29}$$

To calculate $\bar{S}$ we insert into Equation 28 the expression for $\bar{R}$ given by Equation 22 and carry out the integration over τ, t', and s'. The result is

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \iint H(\omega,\underset{\sim}{x}')H^*(\omega,\underset{\sim}{y}')S_0(\omega,\underset{\sim}{x}-\underset{\sim}{x}',\underset{\sim}{y}-\underset{\sim}{y}')d\underset{\sim}{x}'d\underset{\sim}{y}' \quad , \tag{30}$$

where we have defined

$$H(\omega,\underset{\sim}{x}) = \int G(t,\underset{\sim}{x})e^{i\omega t}dt \quad , \tag{31}$$

$$S_0(\omega,\underset{\sim}{x},\underset{\sim}{y}) = \int R_0(\tau,\underset{\sim}{x},\underset{\sim}{y})e^{i\omega\tau}d\tau \quad , \tag{32}$$

and the symbol $(\ )^*$ denotes a complex conjugate. Similarly, an expression for $\tilde{S}$ is obtained by substituting the formula for $\tilde{R}$ given by Equation 23 into Equation 29 and carrying out the integration over τ, t', s', t'', and s''. This procedure yields

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = (2\varepsilon^2/\pi c_0^4)\int\cdots\int H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}')$$

$$\times\ [Z(\omega,\underset{\sim}{y}-\underset{\sim}{x}+\underset{\sim}{x}'-\underset{\sim}{y}')*\omega^4 H(\omega,\underset{\sim}{x}'')H^*(\omega,\underset{\sim}{y}'')$$

$$\times\ S_0(\omega,\underset{\sim}{x}-\underset{\sim}{x}'-\underset{\sim}{x}'',\underset{\sim}{y}-\underset{\sim}{y}'-\underset{\sim}{y}'')]d\underset{\sim}{x}'d\underset{\sim}{y}'d\underset{\sim}{x}''d\underset{\sim}{y}'' \ , \tag{33}$$

where H_0 and Z are defined by

$$H_0(\omega,\underset{\sim}{x}) = \int G_0(t,\underset{\sim}{x})e^{i\omega t}dt \ , \tag{34}$$

$$Z(\omega,\underset{\sim}{\xi}) = \int \Gamma(\tau,\underset{\sim}{\xi})e^{i\omega\tau}d\tau \ . \tag{35}$$

The notation $(\ ) * (\ )$ in Equation 33 denotes a convolution with respect to ω; i.e.,

$$f*g(\omega) = \int f(\omega-\omega')g(\omega')d\omega' \ .$$

(Whenever the convolution symbol appears inside brackets, as in Equation 33, it is to be understood that only the terms inside the brackets are involved in the convolution.) In deriving Equation 33 we have made use of some known results relating the Fourier transform of a product of two functions to the convolution of the transformed functions.

The formulas for R and S given above are accurate to order ε^2; i.e., the error in them is of order ε^3. This is a consequence of the dropping of terms of order ε^3 in Equation 7 and ε^2 in Equation 8 (note that $\tilde{u}$ is of order ε), and the vanishing of the cross terms in Equation 20.

For practical purposes it is usually convenient to assume that all processes under consideration are ergodic, as well as stationary, in time, in which case the average denoted by $\langle \ \rangle$ can be regarded as a time average.

The analysis given above can be generalized; i.e., instead of starting with Equation 1 we can state with Equation 3 and assume that the operator L_0 is determinate with a known inverse and that the operators L_1, L_2, etc. are random with known statistics. The operators L_0, L_1, L_2, etc. need not be otherwise specified. Formulas for the correlation and spectral density functions, analogous to Equations 21-23, 27, 30, and 33, can then be derived for various cases, depending on the additional assumptions made regarding the operators L_0, L_1, L_2, etc. A general analysis of this type has been carried out and is available in report form[19].

In order to proceed further it is necessary to calculate the Green's functions G_0 and G and the transforms H_0 and H. The function G_0, which corresponds to a spherical pulsed wave propagating in a uniform medium, is obtained by solving Equation 10, with L_0 given by Equation 4, subject to the initial condition $G_0 = 0$ for $t < 0$. This yields the familiar waveform given by

$$G_0(t,\underset{\sim}{x}) = (4\pi x)^{-1} \delta(t-c_0^{-1} x) \quad . \tag{36}$$

By inserting the expression for G_0 given by Equation 36 into Equation 14, carrying out the integration over t', and changing the spatial integration variable, we can express the operator L_0^{-1} in the form

$$L_0^{-1}w(t,\underset{\sim}{x}) = (4\pi)^{-1} \int \xi^{-1} w(t-c_0^{-1}\xi, \underset{\sim}{x} + \underset{\sim}{\xi}) d\underset{\sim}{\xi} \quad . \tag{37}$$

The function $H_0(\omega,\underset{\sim}{x})$ is easily obtained by transforming Equation 36 according to Equation 34. The result is

$$H_0(\omega,\underset{\sim}{x}) = (4\pi x)^{-1} e^{ikx} \quad , \tag{38}$$

where $k = \omega/c_0$.

The function $G(t,\underset{\sim}{x})$ is determined by Equation 11, where the operator M is given by Equation 9. With the aid of these equations, along with Equations 4, 5, 6, and 37, we can write the equation for G in the form

$$(c_0^{-2}\partial_t^2-\nabla^2)G(t,\underset{\sim}{x}) + \varepsilon^2\{3c_0^{-2}G_{tt}-(\pi c_0^4)^{-1}$$

$$\times \int \xi^{-1}[\Gamma(c_0^{-1}\xi,\underset{\sim}{\xi})G_{tttt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi}) - 2\Gamma_\tau(c_0^{-1}\xi,\underset{\sim}{\xi})$$

$$\times \; G_{ttt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi}) + \Gamma_{\tau\tau}(c_0^{-1}\xi,\underset{\sim}{\xi})G_{tt}(t-c_0^{-1}\xi,\underset{\sim}{x}+\underset{\sim}{\xi})]d\underset{\sim}{\xi}\}$$

$$= \delta(t)\delta(\underset{\sim}{x}) \quad , \tag{39}$$

where the letter subscripts denote derivatives. The initial condition for G is that $G = 0$ for $t < 0$.

The procedure by which Equation 39 is solved for $G(t,\underset{\sim}{x})$ is described in Appendix A. Since we wish only to calculate the function S, we need only the transform $H(\omega,\underset{\sim}{x})$ of $G(t,\underset{\sim}{x})$, as defined by Equation 31. For the case in which the medium is isotropic [i.e., when $\Gamma(\tau,\underset{\sim}{\xi}) = \Gamma(\tau,\xi)$] this quantity is given by

$$H(\omega,\underset{\sim}{x}) = C(k)\;(4\pi x)^{-1}e^{i\kappa x} \quad , \tag{40}$$

where

$$\kappa = k\{1+\tfrac{1}{2}\varepsilon^2 \,[3+4k^{-1} \int_0^\infty e^{ik\xi}\, \chi(k,\xi) \sin k\xi \, d\xi]\} \ , \tag{41}$$

$$\chi(k,\xi) = k^2\Gamma(c_0^{-1}\xi,\xi) - 2ikc_0^{-1}\Gamma_\tau(c_0^{-1}\xi,\xi) - c_0^{-2}\Gamma_{\tau\tau}(c_0^{-1}\xi,\xi) \ , \tag{42}$$

$$C(k) = 1 + 2\varepsilon^2\psi(k) \ , \tag{43}$$

and

$$\psi(k) = \int_0^\infty e^{ik\xi}\chi(k,\xi) \left[\cos k\xi - \frac{\sin k\xi}{k\xi}\right] \xi \, d\xi \ . \tag{44}$$

In deriving Equations 40-44 higher-order terms in ε have been dropped.

The source term f is assumed to represent a point source in space but one which is random in time. Accordingly we write

$$f(t,\underset{\sim}{x}) = g(t)\delta(\underset{\sim}{x}) \ , \tag{45}$$

where $g(t)$ is a stationary random function with zero mean. Equation 25 then yields

$$R_0(\tau,\underset{\sim}{x},\underset{\sim}{y}) = P_0(\tau)\delta(\underset{\sim}{x})\delta(\underset{\sim}{y}) \ , \tag{46}$$

where $P_0(\tau)$ is defined by

$$P_0(\tau) = \langle g(t)g(t-\tau)\rangle_B \ . \tag{47}$$

Upon transforming Equation 46 according to Equation 32 we obtain

$$S_0(\omega,\underset{\sim}{x},\underset{\sim}{y}) = Q_0(\omega)\delta(\underset{\sim}{x})\delta(\underset{\sim}{y}) \quad , \tag{48}$$

where Q_0 is the transform of P_0; i.e.,

$$Q_0(\omega) = \int P_0(\tau)e^{i\omega\tau}d\tau \quad . \tag{49}$$

Expressions for $\bar{S}$ and $\tilde{S}$ can now be obtained by substituting the formula for S_0 given by Equation 48 into Equations 30 and 33 and carrying out the integration over $\underset{\sim}{x}'$ and $\underset{\sim}{y}'$ in Equation 30 and over $\underset{\sim}{x}''$ and $\underset{\sim}{y}''$ in Equation 33. The result is

$$\bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = Q_0(\omega)H(\omega,\underset{\sim}{x})H^*(\omega,\underset{\sim}{y}) \quad , \tag{50}$$

$$\tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{y}) = (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}')$$

$$\times\ [Z(\omega,\underset{\sim}{y}-\underset{\sim}{x}+\underset{\sim}{x}'-\underset{\sim}{y}')\star\omega^4Q_0(\omega)H(\omega,\underset{\sim}{x}-\underset{\sim}{x}')H^*(\omega,\underset{\sim}{y}-\underset{\sim}{y}')]d\underset{\sim}{x}'d\underset{\sim}{y}' \quad . \tag{51}$$

The spectral density function $S(\omega,\underset{\sim}{x},\underset{\sim}{y})$ can now be calculated in terms of known functions with the aid of Equations 27, 50, 51, 38, and 40.

The expression for $\bar{S}$ and $\tilde{S}$ given by Equations 50 and 51 can be considerably simplified in the case of high-frequency waves; i.e., when the characteristic frequency of the source is much greater than that of the medium. In considering this case we shall restrict our attention to the power spectrum $Q(\omega,\underset{\sim}{x})$, which is defined by

$$Q(\omega,\underset{\sim}{x}) = S(\omega,\underset{\sim}{x},\underset{\sim}{x}) \quad . \tag{52}$$

From Equation 27 we have

$$Q(\omega,\underset{\sim}{x}) = \bar{Q}(\omega,\underset{\sim}{x}) + \tilde{Q}(\omega,\underset{\sim}{x}) \quad , \tag{53}$$

where

$$\bar{Q}(\omega,\underset{\sim}{x}) \equiv \bar{S}(\omega,\underset{\sim}{x},\underset{\sim}{x}) \quad , \tag{54}$$

$$\tilde{Q}(\omega,\underset{\sim}{x}) \equiv \tilde{S}(\omega,\underset{\sim}{x},\underset{\sim}{x}) \quad . \tag{55}$$

Equations 50 and 51 yield

$$\bar{Q}(\omega,\underset{\sim}{x}) = Q_0(\omega) \; |H(\omega,\underset{\sim}{x})|^2 \quad , \tag{56}$$

$$\begin{aligned}\tilde{Q}(\omega,\underset{\sim}{x}) = {} & (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{y}') \\ & \times [Z(\omega,\underset{\sim}{x}'-\underset{\sim}{y}')*\omega^4 Q_0(\omega)H(\omega,\underset{\sim}{x}-\underset{\sim}{x}')H^*(\omega,\underset{\sim}{x}-\underset{\sim}{y}')]d\underset{\sim}{x}'d\underset{\sim}{y}' \quad .\end{aligned} \tag{57}$$

After changing the integration variables in Equation 57 we can write

$$\begin{aligned}\tilde{Q}(\omega,\underset{\sim}{x}) = {} & (2\varepsilon^2/\pi c_0^4)\iint H_0(\omega,\underset{\sim}{x}-\underset{\sim}{x}')H_0^*(\omega,\underset{\sim}{x}-\underset{\sim}{x}'') \\ & \times [Z(\omega,\underset{\sim}{x}''-\underset{\sim}{x}')*\omega^4 Q_0(\omega)H(\omega,\underset{\sim}{x}')H^*(\omega,\underset{\sim}{x}'')]d\underset{\sim}{x}'d\underset{\sim}{x}'' \quad .\end{aligned} \tag{58}$$

The first step in the high-frequency analysis is to obtain an asymptotic expansion, valid for large k, for the quantity κ. This is easily accomplished by integrating by parts in Equation 41, after substituting for χ from Equation 42. This yields the approximation

$$\kappa \simeq k + i\alpha \,, \tag{59}$$

where

$$\alpha = \varepsilon^2 k^2 \ell \quad . \tag{60}$$

The quantity ℓ is a characteristic length scale associated with the medium, and is defined by

$$\ell = \int_0^\infty \Gamma(c_0^{-1}\xi, \xi)\, d\xi \ . \tag{61}$$

With the aid of Equations 38, 40, and 59 we can write Equations 56 and 58 in the form

$$\bar{Q}(\omega, \underset{\sim}{x}) = Q_0(\omega)\left|C(k)^2\right|(4\pi x)^{-2} e^{-2\alpha x} \quad , \tag{62}$$

$$\tilde{Q}(\omega, \underset{\sim}{x}) = \frac{2\varepsilon^2}{\pi(4\pi)^4} \iint \frac{e^{ik(|\underset{\sim}{x}-\underset{\sim}{x}'| - |\underset{\sim}{x}-\underset{\sim}{x}''|)}}{|\underset{\sim}{x}-\underset{\sim}{x}'|\, x' \ |\underset{\sim}{x}-\underset{\sim}{x}''|\, x''}$$

$$\times \left[Z(\omega, \underset{\sim}{x}'' - \underset{\sim}{x}') * k^4 |C(k)|^2 Q_0(\omega) e^{ik(x'-x'')} e^{-\alpha(x'+x'')} \right] d\underset{\sim}{x}'\, d\underset{\sim}{x}'' \quad , \tag{63}$$

where, from Equation 43,

$$|C(k)|^2 = 1 + 4\varepsilon^2 \mathrm{Re}\psi(k) \quad . \tag{64}$$

(In deriving Equation 64 terms of order ε^4 were dropped.)

The integral over $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$ in Equation 63 has been evaluated using the forward-scatter approximation. The details of that calculation are given in Appendix B. The resulting approximate expression for $\tilde{Q}(\omega,\underset{\sim}{x})$ can be written

$$\tilde{Q}(\omega,\underset{\sim}{x}) = (4\pi x)^{-2}[W(\omega)*|C(k)|^2(1-e^{-2\alpha x})Q_0(\omega)] \ , \tag{65}$$

where W is defined by

$$W(\omega) = (4\pi\ell)^{-1}\ \hat{Z}(\omega,k) \quad , \tag{66}$$

and

$$\hat{Z}(\omega,\nu) = 2\int_0^\infty Z(\omega,\xi)\cos\nu\xi\ d\xi \ . \tag{67}$$

An expression for $Q(\omega,\underset{\sim}{x})$ can now be obtained by substituting the formulas for $\bar{Q}(\omega,\underset{\sim}{x})$ and $\tilde{Q}(\omega,\underset{\sim}{x})$ given by Equations 62 and 65 into Equation 53. In so doing we simplify matters slightly by making the approximation $|C(k)|^2 = 1$. After dividing through by the spherical-spreading term $(4\pi x)^{-2}$ we obtain finally

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = e^{-2\alpha x}Q_0(\omega) + W(\omega)*(1-e^{-2\alpha x})Q_0(\omega) \quad . \tag{68}$$

It should be pointed out here that, although the error in the general formulas for R and S given by Equations 21, 22, 23, 27, 30, and 33 is of order ε^3, the error in Equation 68 is of order ε^2. This is because some terms of order ε^2 were dropped in the derivation of this equation.

We see from Equation 68 that, as $\alpha x \to 0$,

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) \to Q_0(\omega) \quad .$$

Thus, as ε and/or x (the source-field point distance) goes to zero, the wave spectrum (with the spherical-spreading term factored out) approaches the source spectrum, as we would expect. In the opposite limit, i.e., as $\alpha x \to \infty$, Equation 68 shows that

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) \to W{*}Q_0(\omega) \quad .$$

We see therefore that the wave spectrum (again apart from the spherical-spreading term) tends to a limiting form as $x \to \infty$. This limiting form, which is given by the convolution of W with Q_0, is referred to here as the fully-developed spectrum.

It may be verified by direct integration of Equation 68 that

$$(4\pi x)^2 \int Q(\omega,\underset{\sim}{x})\,d\omega = \int Q_0(\omega)\,d\omega \quad . \tag{69}$$

In the derivation of Equation 69 we have used the fact that

$$\int W(\omega)\,d\omega = 1 \quad .$$

Equation 69 shows that the total signal power; i.e., the area under the spectral curve, normalized by the spherical-spreading term, is conserved.

We can simplify Equation 68 further by assuming a narrow-band source, i.e., by assuming that the characteristic width of the source spectrum $Q_0(\omega)$ is much less than that of the function $W(\omega)$. Then, insofar as the convolution integral is concerned, $Q_0(\omega)$ can be regarded as a delta function. Accordingly we replace $Q_0(\omega)$ in the convolution term by

$$A_0[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)]$$

(since Q_0 must be an even function), where $\omega_0 > 0$ (ω_0 is called the carrier frequency) and $A_0 > 0$. We can also write, in this case,

$$e^{-2\alpha x} Q_0(\omega) \simeq e^{-2\alpha_0 x} Q_0(\omega) \quad ,$$

where $\alpha_0 = \varepsilon^2 k_0^2 \ell$ and $k_0 = \omega_0/c_0$. Then Equation 68 becomes

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = e^{-2\alpha_0 x} Q_0(\omega) + (1 - e^{-2\alpha_0 x}) Q_\infty(\omega) \quad , \tag{70}$$

where

$$Q_\infty(\omega) = A_0[W(\omega-\omega_0) + W(\omega+\omega_0)] \quad . \tag{71}$$

By introducing a "broadening parameter" β, defined by

$$\beta = 1 - e^{-2\alpha_0 x} \quad , \tag{72}$$

we can write Equation 70 in the form

$$(4\pi x)^2 Q(\omega,\underset{\sim}{x}) = (1-\beta) Q_0(\omega) + \beta Q_\infty(\omega) \quad . \tag{73}$$

Thus we see that the wave spectrum (with the spherical-spreading term factored out) can be regarded in this case as a linear (in β) interpolation between the source spectrum $Q_0(\omega)$ and the fully-developed spectrum $Q_\infty(\omega)$.

We define the bandwidth $\tilde{\omega}$ of the wave spectrum for the narrow-band case by writing

$$\tilde{\omega}(x) = \left[\int_0^\infty (\omega-\omega_0)^2 Q(\omega,\underset{\sim}{x})\, d\omega \Big/ \int_0^\infty Q(\omega,\underset{\sim}{x})\, d\omega\right]^{\frac{1}{2}} \quad . \tag{74}$$

By substituting the expression for Q given by Equation 73 into Equation 74 we obtain

$$\tilde{\omega}(x) = [(1-\beta)\tilde{\omega}_0^2 + \beta\tilde{\omega}_\infty^2]^{\frac{1}{2}} \quad , \tag{75}$$

where $\tilde{\omega}_0$ is the bandwidth of the source spectrum and $\tilde{\omega}_\infty$ is the bandwidth of the fully-developed spectrum; i.e.,

$$\tilde{\omega}_0 = \left[\int_0^\infty (\omega-\omega_0)^2 Q_0(\omega)\, d\omega \Big/ \int_0^\infty Q_0(\omega)\, d\omega\right]^{\frac{1}{2}} \quad , \tag{76}$$

$$\tilde{\omega}_\infty = \left[\int_0^\infty (\omega-\omega_0)^2 Q_\infty(\omega)\, d\omega \Big/ \int_0^\infty Q_\infty(\omega)\, d\omega\right]^{\frac{1}{2}} \quad . \tag{77}$$

Equation 75 shows that $\tilde{\omega}(x)$ increases monotonically with x from the value $\tilde{\omega}_0$ at $x = 0$, and that it approaches $\tilde{\omega}_\infty$ with increasing x.

If we assume that $\tilde{\omega}_0 \simeq 0$, which is consistent with the assumption of a narrow-band source, then Equation 75 yields

$$\tilde{\omega}(x) \simeq \beta^{\frac{1}{2}}\, \tilde{\omega}_\infty \quad . \tag{78}$$

When $\alpha_0 x \ll 1$ we have, from Equation 72, $\beta \simeq 2\alpha_0 x = 2\varepsilon^2 k_0^2 \ell x$, and hence, from Equation 78,

$$\tilde{\omega}(x) \simeq \varepsilon k_0\, (2\ell x)^{\frac{1}{2}}\, \tilde{\omega}_\infty \quad . \tag{79}$$

Equation 79 is valid when $\epsilon k_0(2\ell x)^{\frac{1}{2}} << 1$, i.e., when $\tilde{\omega} << \tilde{\omega}_\infty$. This equation shows that, when the propagation distance is small, the spectral bandwidth increases as the square root of the propagation distance, and is also linear in the carrier frequency in this range.

In order to show the broadening phenomenon graphically, numerical calculations of the quantity $(4\pi x)^2 Q(\omega,\underset{\sim}{x})$ as a function of ω have been made for various values of β using Equation 73. For this purpose the source spectrum $Q_0(\omega)$ was chosen to be a narrow-band Gaussian function, centered at $\omega = \omega_0$ and reflected about the $\omega = 0$ axis (since Q_0 must be an even function). The function $W(\omega)$ was also chosen to be a reflected Gaussian, with a maximum in the first instance at $\omega = 0$ and in the second at a frequency Ω_0 for which $0 < \Omega_0 << \omega_0$.

The results of these calculations are plotted (in dimensionless coordinates) in Figures 1 and 2. All of the curves in each figure are plotted on the same scale. Note that in each figure the curve labeled $\beta = 0$ corresponds to the source spectrum, the curve labeled $\beta = 1$ corresponds to the fully-developed spectrum, and those labeled with values of β between zero and one correspond to intermediate stages in the broadening process. Both sets of curves show clearly the broadening of the wave spectrum with increasing β, which is equivalent to increasing either the propagation distance or the strength of the randomness of the medium. The two sets differ, however, in one respect. The results shown in Figure 2, for which the function $W(\omega)$ has a maximum at a non-zero value of ω, are marked by the appearance of side lobes on the broadened spectrum. In Figure 1, by contrast, for which the maximum of $W(\omega)$ occurs at $\omega = 0$, no such side lobes appear.

The results obtained here appear generally to be in qualitative agreement with observations, as can be seen by, for example, comparing Figure 1 with Figure 11 of Reference 1 or Figure 3 of Reference 2. Note moreover that the observations reported in References 1 and 2 indicate conservation of total signal power, which is also consistent with the present results (cf. Equation 69).

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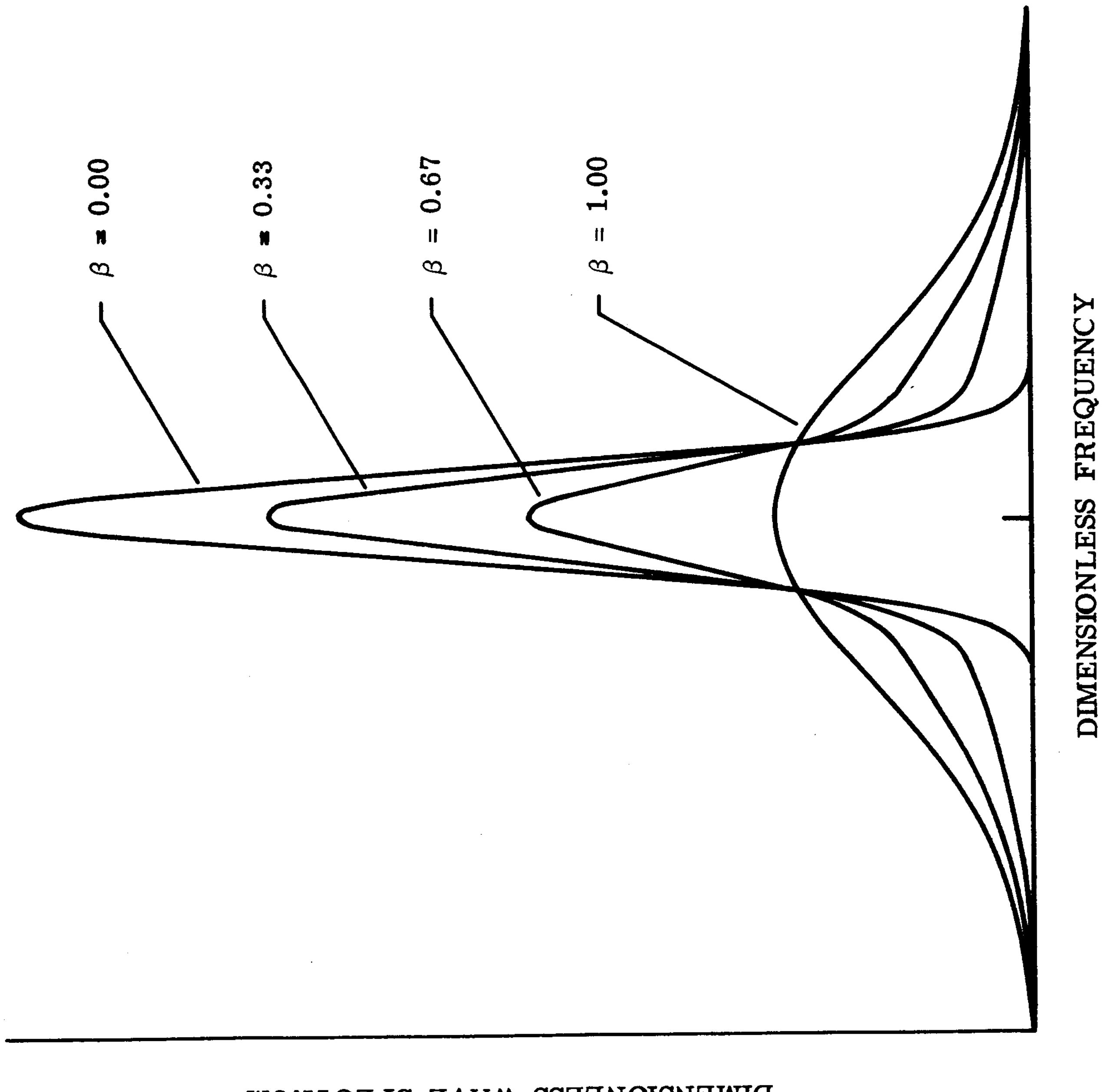
LIST OF FIGURES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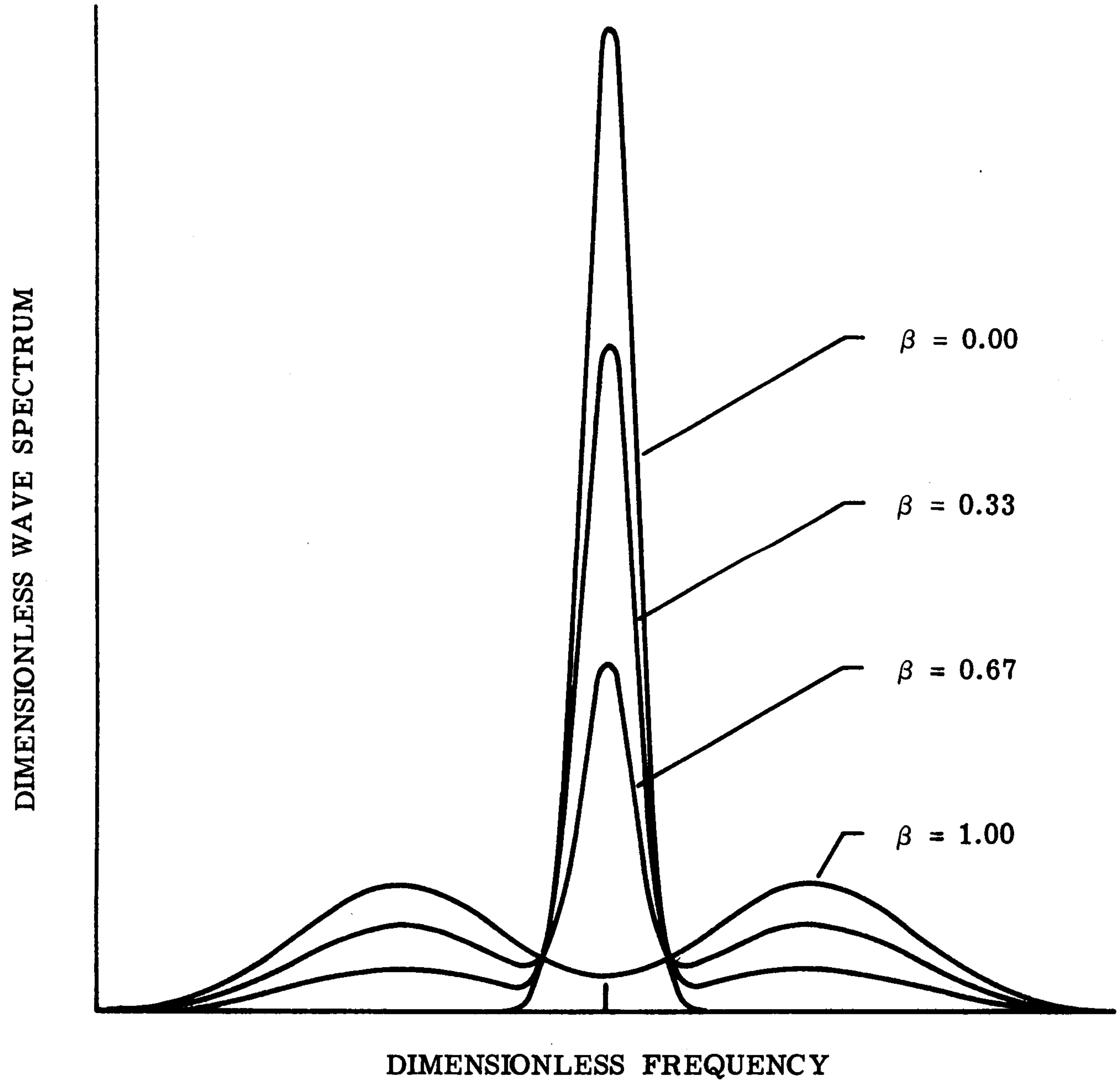
β = 0.33
β = 0.67
β = 1.00
DIMENSIONLESS FREQUENCY
DIMENSIONLESS WAVE SPECTRUM

Fig. 1



β = 0.00
β = 0.33
β = 0.67
β = 1.00
DIMENSIONLESS WAVE SPECTRUM
DIMENSIONLESS FREQUENCY

Fig. 2

APPENDIX A. CALCULATION OF $G(t,\underset{\sim}{x})$ AND $H(\omega,\underset{\sim}{x})$

The function $G(t,\underset{\sim}{x})$ is determined by Equation 39, together with the initial condition $G = 0$ for $t < 0$. An equation for the transform $H(\omega,\underset{\sim}{x})$ of $G(t,\underset{\sim}{x})$, as defined by Equation 31, is obtained by transforming both sides of Equation 39. The result is

$$\left[\nabla^2 + (1+3\varepsilon^2)k^2 \right] H(\omega,\underset{\sim}{x}) + (\varepsilon^2 k^2/\pi) \int \xi^{-1} e^{ik\xi}$$

$$\times \; \chi(k,\underset{\sim}{\xi}) H(\omega,\underset{\sim}{x}+\underset{\sim}{\xi}) d\underset{\sim}{\xi} = -\delta(\underset{\sim}{x}) \quad , \tag{A1}$$

where the function $\chi(k,\underset{\sim}{\xi})$ is defined by

$$\chi(k,\underset{\sim}{\xi}) = k^2 \Gamma(c_0^{-1}\xi,\underset{\sim}{\xi}) - 2ikc_0^{-1}\Gamma_\tau(c_0^{-1}\xi,\underset{\sim}{\xi})$$

$$- c_0^{-2}\Gamma_{\tau\tau}(c_0^{-1}\xi,\underset{\sim}{\xi}) \quad . \tag{A2}$$

In order to solve Equation A1 we introduce the spatial Fourier transform $\hat{H}(\omega,\underset{\sim}{m})$ of $H(\omega,\underset{\sim}{x})$, defined by

$$\hat{H}(\omega,\underset{\sim}{m}) = \int H(\omega,\underset{\sim}{x}) e^{-i\underset{\sim}{m}\cdot\underset{\sim}{x}} d\underset{\sim}{x} \quad , \tag{A3}$$

where $\underset{\sim}{m} \cdot \underset{\sim}{x} \equiv \sum_{i=1}^{3} m_i x_i$. Transforming both sides of Equation A1 according to the prescription given by Equation A3 and solving for $\hat{H}$ yields

$$\hat{H}(\omega,\underset{\sim}{m}) = [D(k,\underset{\sim}{m})]^{-1} \quad , \tag{A4}$$

where we have defined

$$D(k,\underset{\sim}{m}) = m^2 - (1+3\varepsilon^2)k^2 - (\varepsilon^2 k^2/\pi)\int \xi^{-1} e^{ik\xi}$$

$$\times\ \chi(k,\underset{\sim}{\xi}) e^{i\underset{\sim}{m}\cdot\underset{\sim}{\xi}}\, d\underset{\sim}{\xi} \quad . \tag{A5}$$

With the aid of Equation A4 we can now express $H(\omega,\underset{\sim}{x})$ as an inverse transform, i.e., we write

$$H(\omega,\underset{\sim}{x}) = (8\pi^3)^{-1}\int [D(k,\underset{\sim}{m})]^{-1} e^{i\underset{\sim}{m}\cdot\underset{\sim}{x}} d\underset{\sim}{m} \quad . \tag{A6}$$

In order to proceed further we assume that the medium is statistically isotropic, so that we can write $\Gamma(\tau,\underset{\sim}{\xi}) = \Gamma(\tau,\xi)$. Then, in view of Equation A2, we can also write $\chi(k,\underset{\sim}{\xi}) = \chi(k,\xi)$. The angular integration in Equation A5 can now be carried out, yielding

$$D(k,\underset{\sim}{m}) = D(k,m) = m^2 - (1+3\varepsilon^2)k^2 - 4\varepsilon^2 k^2 m^{-1} \int_0^\infty e^{ik\xi}\chi(k,\xi)\,\sin m\xi\, d\xi \ . \tag{A7}$$

Upon substituting the expression for D given by Equation A7 into Equation A6 and carrying out the angular integration we find that

$$H(\omega,\underset{\sim}{x}) = (2\pi^2 x)^{-1} \int_0^\infty [D(k,m)]^{-1} m\,\sin mx\, dm \quad . \tag{A8}$$

The integral in Equation A8 can be evaluated by means of contour integration, after which the expression for H can be written

$$H(\omega,\underset{\sim}{x}) = (2\pi x)^{-1}[D_m(k,\kappa)]^{-1}\kappa e^{i\kappa x} \quad . \tag{A9}$$

Here D_m denotes the derivative of $D(k,m)$ with respect to m (regarded now as a complex variable), and κ is the root of the dispersion equation $D(k,\kappa) = 0$ which has the property that $\kappa \to k$ as $\varepsilon \to 0$. This root is given, to lowest order in ε, by Equation 41. Upon substituting the expression for κ given by Equation 41 into Equation A9, after calculating D_m using Equation A7, we obtain the expression for H given by Equation 40.

The function $G(t,\underset{\sim}{x})$ can now be obtained by applying the inverse Fourier transform to Equation 40. We shall not carry out that calculation here, however, since we need only the function $H(\omega,\underset{\sim}{x})$. That calculation was carried out in Reference 20 for the case of a time-independent medium.

APPENDIX B. CALCULATION OF $\tilde{Q}$ USING THE FORWARD-SCATTER APPROXIMATION

By making explicit the ω' integration in Equation 63 and changing the order of integration we can write the expression for $\tilde{Q}$ in the form

$$\tilde{Q}(\omega,\underset{\sim}{x}) = [2\varepsilon^2/\pi(4\pi)^4] \int k'^4 |C(k')|^2 \, Q_0(\omega') I(\omega,\underset{\sim}{x};\omega') d\omega' \quad , \qquad \text{(B1)}$$

where the integral I is defined by

$$I(\omega,\underset{\sim}{x};\omega') = \iint \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'| - |\underset{\sim}{x}-\underset{\sim}{x}''|)}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'|\underset{\sim}{x}-\underset{\sim}{x}''|x''} Z(\omega-\omega',\underset{\sim}{x}''-\underset{\sim}{x}')$$

$$\times \, e^{ik'(x'-x'')} e^{-\alpha'(x'+x'')} d\underset{\sim}{x}' d\underset{\sim}{x}'' \quad . \qquad \text{(B2)}$$

Here $k' = \omega'/c_0$ and $\alpha' = \varepsilon^2 k'^2 \ell$. Equation B2 can be written in the alternate form

$$I(\omega,x;\omega') = \iint \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'| + x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \; \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''| + x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''| \, x''}$$

$$\times \; Z(\omega-\omega',\underset{\sim}{x}''-\underset{\sim}{x}') e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|} e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times \; e^{-\alpha'(x'+x'')} d\underset{\sim}{x}' d\underset{\sim}{x}'' \quad , \qquad \text{(B3)}$$

which is more convenient for the application of the forward-scatter approximation.

We begin the analysis by substituting for Z in terms of Γ in Equation B3 with the aid of Equation 35. By changing the order of integration in the resulting expression for I we get

$$I = \int e^{i(\omega-\omega')} \iint \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'| + x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \; \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''| + x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''| \; x''}$$

$$\times \; \Gamma(\tau,\underset{\sim}{x}'' - \underset{\sim}{x}') e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|} e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times \; e^{-\alpha'(x'+x'')} d\underset{\sim}{x}' d\underset{\sim}{x}'' d\tau \; . \tag{B4}$$

Next we use Equation 24 to substitute for Γ in terms of μ in Equation B4. Upon reversing the order of the averaging and integration (over $\underset{\sim}{x}'$ and $\underset{\sim}{x}''$) processes, we note that the double spatial integral can be split into a product of two integrals. Equation B4 can then be written

$$I = \int e^{i(\omega-\omega')\tau} \langle J \rangle_A d\tau \quad , \tag{B5}$$

where

$$J = J_+ J_- \quad , \tag{B6}$$

$$J_+ = \int \frac{e^{ik'(|\underset{\sim}{x}-\underset{\sim}{x}'| + x')}}{|\underset{\sim}{x}-\underset{\sim}{x}'|x'} \; \mu(t,\underset{\sim}{x}') e^{i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}'|} e^{-\alpha' x'} d\underset{\sim}{x}' \quad , \tag{B7}$$

and

$$J_- = \int \frac{e^{-ik'(|\underset{\sim}{x}-\underset{\sim}{x}''| + x'')}}{|\underset{\sim}{x}-\underset{\sim}{x}''| \; x''} \; \mu(t-\tau,\underset{\sim}{x}'') e^{-i(k-k')|\underset{\sim}{x}-\underset{\sim}{x}''|}$$

$$\times \; e^{-\alpha' x''} d\underset{\sim}{x}'' \quad . \tag{B8}$$

We can now apply the forward-scatter approximation, as discussed in Reference 21, to the integrals J_+ and J_-. This yields

$$J_+ = (2\pi i/k'x)e^{ik'x}\int_0^x \mu(t,0,0,x')e^{i(k-k')(x-x')}$$

$$\times \quad e^{-\alpha'x'}dx' + O(k'^{-2}) \quad , \qquad \text{(B9)}$$

$$J_- = -(2\pi i/k'x)e^{-ik'x}\int_0^x \mu(t-\tau,0,0,x'')e^{-i(k-k')(x-x'')}$$

$$\times \quad e^{-\alpha'x''}dx'' + O(k'^{-2}) \quad . \qquad \text{(B10)}$$

In the derivation of Equations B9 and B10 we have set $\underset{\sim}{x} = (0,0,x)$. This entails no loss of generality since the medium has been assumed statistically isotropic.

Conditions for the validity of the forward-scatter approximation are given in Reference 21. In the present context these conditions take the form

$$k_1^{-1} \ll x \ll k_1 \ell^2 \quad , \qquad \text{(B11)}$$

where k_1 is a characteristic wavenumber associated with the wave field.

By substituting the expressions for J_+ and J_- given by Equations B9 and B10 into Equation B6, dropping terms of order k'^{-3}, and averaging, we obtain

$$\langle J \rangle_A = (4\pi^2/k'^2x^2) \int_0^x\int_0^x \Gamma(\tau,x''-x')e^{i(k-k')(x''-x')}$$

$$\times \quad e^{-\alpha'(x''+x')}dx'dx'' \quad . \qquad \text{(B12)}$$

The double integral in Equation B12 can be partially evaluated with the aid of the coordinate transformation $\xi = x'' - x'$, $\eta = x'' + x'$. The result is

$$\langle J \rangle_A = (4\pi^2/\alpha' k'^2 x^2) \int_0^x \Gamma(\tau,\xi)(e^{-\alpha'\xi} - e^{-2\alpha' x} e^{\alpha'\xi})$$

$$\times \cos[(k-k')\xi]\, d\xi \quad . \tag{B13}$$

In deriving Equation B13 we have made use of the fact that $\Gamma(\tau,\xi)$ is given in ξ.

We can now get a series expansion for $\langle J \rangle_A$ in powers of α' (which is equivalent to an expansion in powers of ε^2) by expanding the terms $\exp(\alpha'\xi)$ and $\exp(-\alpha'\xi)$ in Equation B13 and integrating term by term. This yields

$$\langle J \rangle_A = (4\pi^2/\alpha' k'^2 x^2) \sum_{n=0}^{\infty} (-1)^n \, [1-(-1)^n e^{-2\alpha' x}](\alpha'^n/n!)$$

$$\times \int_0^x \xi^n \Gamma(\tau,\xi) \cos[(k-k')\xi]\, d\xi \quad . \tag{B14}$$

When $x \gg \ell$ the integration in Equation B14 can be extended to $+\infty$ without introducing significant error into the integral. Upon dropping all but the first term of the resulting expansion we obtain

$$\langle J \rangle_A \simeq (4\pi^2/\alpha' k'^2 x^2)(1-e^{-2\alpha' x}) \int_0^\infty \Gamma(\tau,\xi)\cos[(k-k')\xi]\, d\xi \; . \tag{B15}$$

An approximate expression for the integral I can now be obtained by substituting the result for $\langle J \rangle_A$ given by Equation B15 into Equation B5 and carrying out the integration over τ. This yields

$$I = (2\pi^2/\alpha' k'^2 x^2)(1-e^{-2\alpha' x})\hat{Z}(\omega-\omega', k-k') \quad , \tag{B16}$$

where $\hat{Z}$ is defined by Equation 67. Upon combining Equations B1 and B16 we obtain the expression for $\tilde{Q}$ given by Equation 65.